ARITHMOLOGY

BY

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ARITHMOLOGY;

OR,

THEORY OF COMMON ARITHMETIC,

FULLY PROVED WITHOUT ALGEBRA.

BY

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PREFACE.

In presenting this small Work to the public, I will briefly explain my object, and the reason why I wished not to multiply the innumerable crowd of arithmetic books. As to the last, I must confess, all seem to be written by men, whose abilities could not produce better ones, or who appeared to have considered common arithmetic as a matter of too little consequence to take much notice of it. However, all are compositions of a good quantity of rules, which without perpetual practice must be forgotten, or at least confused, as they cannot be understood, but only learned, and the pupils can only thus be trained as animals, not taught, as intelligent beings ought to be. My object was to give a theory of reckoning, I mean not only rules, but

an understanding and proof of the correctness of them, in order that he, who understands this theory well, may be able not only to find out all similar cases, not mentioned here, but to renew the rules in his mind, if he has forgotten them. To have shown the way, how to reach this object, was my intention; and only then will I be content, to have published a new arithmetic, when it has caused an alteration in a matter of such general importance, and in which the authors have too long continued in their old way, to the greatest injury of their readers.

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London, October, 1844.

NUMERATION.

§ I. To number means either to pronounce a given number, or to write down a dictated one. To do the first, we only require to know, how six numbers are to be pronounced, and this we can do, if we know, that from the right to the left the first figure signifies the units, the second the tens, the third the hundreds, the fourth the thousands, the fifth the tens of thousands, the sixth the hundreds of thousands. If we divide a given number from the right to the left into classes of six figures, we give to each class, except to the first, a peculiar name; that of the second class is "million," of the third "billion," of the fourth "trillion," of the fifth "quadrillion," and so on, "quintillion," "sextillion," "septillion," "octillion," "nonillion," "decillion," &c. Now we can easily pronounce every given number after this rule: we divide the number from the right to the left into classes of six figures,

and pronounce the figures from the left to the end of every class, always adding the name of the class, as soon as we have pronounced the figures of that class. For instance:

456 | 781247 | 156745;

four hundred and fifty-six billions, seven hundred eighty-one thousand two hundred and forty-seven millions, one hundred fifty-six thousand seven hundred and forty-five.

570 | 140005 | 367001 | 020300 | 401237;

five hundred and seventy quadrillions, one hundred and forty thousand and five trillions, three hundred sixty-seven thousand and one billions, twenty thousand and three hundred millions, four hundred and one thousand two hundred and thirty-seven.

To write a dictated number, we must first ascertain, of how many figures the dictated number consists. This we easily find by multiplying the quantity of full classes by six, and adding as many units, as there are figures in the last imperfect class, if there is one. Then we have only occasion to write down the dictated figures; the places of those figures which are not dictated must be filled up with cyphers: for instance—seven hundred fifty thousand and six—750006; three hundred thousand and twenty—

300020; seven hundred and five billions, four thousand eight hundred and six millions, three hundred and nine—705,004806,000309, &c.

The four first Rules.

§. 2. We will only give the definitions, names and signs used in each, and some rules for reckoning by heart, presuming the knowledge of reckoning.

(*a*) Addition. To add means to find a number, called sum, which contains as many units as two or more together; the numbers to be added are called summends, the sign is +, and signifies "plus," that is more, or "and."

(*b*) Subtraction. To subtract one number from another means to take from one number the units of another. The last number is called subtrahend, the former minuend, the result remainder, rest or difference: the sign is —, which means less or minus.

(*c*) Multiplication is nothing but an abbreviated addition; viz., instead of $9+9+9+9$, we can say 4 times 9, which means that the 9 is to be added four times. Therefore to multiply a number means to add it as many times, as another number contains units, or to increase one number as many times, as any other signifies. The

number to be multiplied (9) is called multiplicand, the other, by which the multiplicand is to be multiplied, multiplicator (4); both without any difference factors, the result product; the sign is $\times$ or $\cdot$ "meaning times."

(*d*) Division. To divide a number means to separate it into a given number of equal parts, or to lessen a number a given number of times. The number to be divided is called dividend, the one by which it is divided divisor, the result, quotient, the sign is "$\div$" and means "by." The dividend must always be put before the sign of the division. $=$ means equal.

Some Abbreviations for reckoning by heart.

We add to any number the number 9, by subtracting one from the units, and adding one to the tens; viz., $58+9=67$, $61+9=70$; we add 8 by subtracting 2 from the units, and adding one to the tens; viz., $58+8=66$; $72+8=80$; we add 7 by subtracting 3 from the units, and adding one to the tens; viz., $58+7=65$; $74+7=81$. We subtract 9, 8 and 7 from a number by adding one, two, and three to the units, and subtracting one from the tens; viz., $58-9=49$; $46-9=37$; $56-8=48$; $75-8=67$; $45-7=38$; $62-7=55$, &c. These rules

are not to be used, if we have to subtract 7, 8 or 9 from a number, whose units are so great, that we can subtract 7, 8 or 9 from the units ; at least it would not spare time, but confuse.

The named Numbers.

§ 3. A number expresses a certain quantity of units, without any distinction, of what sort of things they are. A named number we call a number, which signifies a quantity of certain things, as 7 lbs., £4. 5*s.* Those which signify things of the same sort, we call equal named numbers, homonymous, as £3, £5, &c.; but such as express things of the same kind only, not of the same sort, are called homogeneous numbers. The number, which expresses, how many things one sort contains of things of the same kind, but of another sort, is called the number of reduction ; viz., from pounds to shillings, the reduction-number is 20, because £1=20*s.*; from lbs. to oz., the reduction-number is 16, because 1 lb. contains 16 oz.

To resolve a given named number means, to alter it into another number, which signifies things of a lower quality, but having collectively the same value. To resolve £2 to shillings is, to change £2 into a number, which expresses,

how many shillings are contained in £2, and this we find by multiplying the given number by the reduction-number; therefore £2 = to 40*s.*, 4 lb. to 64 oz. To reduce a given number means to alter it into such a number as signifies things of a higher quality, but having collectively the same value; therefore to reduce 36*d.* into shillings, means to change 36*d.* into such a number, as signifies, how many shillings are equal to 36*d.*; and this we find by dividing the given number by the reduction-number: viz., 36*d.*=3*s.*; 32 oz.=2 lb., &c.

Remark.—Only homonymous numbers can be added, subtracted and divided.

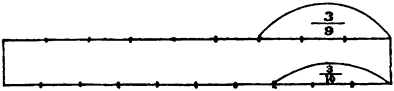
Fractions.

§ 4. If we divide one or more whole things into a certain quantity of parts, the term used for one or more parts is called fraction. A fraction therefore is the expression of one or more equal parts, into which one or more whole things are divided. It is expressed by two numbers, one signifying the number of parts taken, and called numerator; the other signifies the number of parts, into which the whole is divided, and is called denominator; the numerator is to be written above, the denominator

under a straight line; viz., if we divide a straight line into 10 parts, and take 3 of those parts, we should express it by $\frac{3}{10}$, 3 counts the parts taken, 10 signifies the number of parts, into which the whole is divided. Homonymous fractions are those which have the same denominator. Proper fractions are those, of which the denominator is greater than the numerator; improper are those, of which the denominator is smaller than the numerator.

§ 5. From the definition before given, it is obvious that the value of a fraction increases, if the numerator increases, or the denominator decreases. For if the numerator increases, the value of the parts does not increase, but more of them are taken, and the greater is the whole fraction. If the denominator decreases, the number of parts, into which the whole is divided, decreases, but every part must increase, therefore, the whole fraction. The whole fraction must decrease, when the numerator decreases, or the denominator increases, for in the former case the number of parts taken decreases, in the latter each *separate* part decreases, therefore in both cases the whole fraction decreases. For instance, if we divide a straight line into 10 parts, and take 3 of them, the signification of it $\frac{3}{10}$

would be a fraction; $\frac{4}{10}$ would obviously be greater than $\frac{3}{10}$; $\frac{3}{9}$ greater than $\frac{3}{10}$, because in the first fraction there are three parts, each of which is a part of a line divided into 9 parts; in the second are 3 parts, each of which is a part of a line divided into 10 parts, since $\frac{1}{10}$ is evidently smaller than $\frac{1}{9}$, it is clear that $\frac{3}{9}$ are greater than $\frac{3}{10}$. That $\frac{4}{10}$ is greater than $\frac{3}{10}$ is obvious; as 10 parts taken must be more than 3.



§ 6. A fraction is multiplied into a number by dividing the number into the denominator, or if we cannot do this without a remainder, by multiplying the numerator by this number; viz., $\frac{5}{12} \times 4 = \frac{5}{3}$; $\frac{5}{12} \times 5 = \frac{25}{12}$. For if we divide the denominator by the number, the denominator decreases so many times, as that number contains units, which means that the whole is divided into so many parts less, therefore each part becomes so many times greater, that is, the whole fraction is so many times greater, or it is by that number multiplied. If we multiply the numerator, it is obvious, that the

number of parts taken increases, therefore, the whole fraction multiplied.

§ 7. A number is divided into a fraction by dividing the number into the numerator, or multiplying the denominator by that number; viz., $\frac{8}{9} \div \frac{4}{9} =$; $\frac{8}{9} \div 3 = \frac{8}{27}$. For if we divide the numerator, the number of parts taken is obviously diminished; therefore the whole fraction, that is by that number divided. If we multiply the denominator by that number, the denominator of course increases, that is, the number of parts into which the whole is divided; therefore each part decreases, that is, the whole fraction is by that number divided.

§ 8. From § 6 and 7 it follows, that we can change every fraction in the two following methods, without changing its value; viz., we can multiply and divide the numerator and the denominator by the same number; for instance, $\frac{6}{8} = \frac{12}{16}$ (numerator and denominator multiplied by 2,) $\frac{6}{8} = \frac{3}{4}$ (numerator and denominator divided by 2.)

1. For, if we multiply the numerator, the whole fraction is multiplied (after § 6); if we multiply the denominator, the whole fraction is divided (after § 7), and in both cases by the same number; but if any number is multiplied

and divided by the same number, the value of it cannot be altered.

2. If we divide the numerator by a number we divide the whole fraction (§ 7); if we divide the denominator, we multiply the whole fraction (§ 6); therefore the fraction cannot be altered in its value.

§ 9. It will not be superfluous to give here some rules for ascertaining, if we can divide a number by another of one figure, without any remainder or not. A number can be divided without any remainder:

(*a*) By 2, if the units are even; viz., 234, 67818, &c.

(*b*) By 3, if the sum of the figures can be divided by 3, without any remainder; viz., 672312, for, $6+7+2+3+1+2=21$, $21 \div 3=7$; 8147655, for $8+1+4+7+6+5+5=36$, $36 \div 3=12$.

(*c*) By 4, if the units and tens, considered together as a number, can be divided by 4, without a remainder; viz., 672324, because $24 \div 4=6$; 7318323140, because $40 \div 4=10$.

(*d*) By 5, if the units are either 0 or 5; viz., 1005, 6050.

(*e*) By 6, if the units are even, and the sum of the figures can be divided by 3, without any remainder; viz., 314532, because 2 is

an even number, and, $3+1+4+5+3+2=18$, $18\div3=6$.

(*f*) For 7 there is not any rule.

(*g*) For 8 the rule is not important, for the units, tens and hundreds must be divided by 8, without any remainder; viz., 367360, for $360\div8=45$.

(*h*) By 9, if the sum of the figures can be divided by 9, without a remainder; viz., 1931625, for $1+9+3+1+6+2+5=27$; $27\div9=3$.

§ 10. Since $\frac{4}{1}\div5=\frac{4}{5}$ (§ 7), we can say $\frac{4}{5}=\frac{4}{1}\div5$, that is, $\frac{4}{5}=4\div5$. Expressed in words it is this, that a fraction is nothing else than an example of division, in which the dividend becomes the numerator, and the divisor the denominator; viz., $\frac{8}{2}=8\div2=4$; now we see, that we can divide one number into another, even when the divisor is greater than the dividend, for the result or quotient is a proper fraction; viz., $2\div5=\frac{2}{5}$.

§ 11. Two inhomonymous fractions are made homonymous, by multiplying the numerator and denominator of each by the denominator of the other; viz., $\frac{2}{3}$, $\frac{4}{5}$, are made homonymous, thus, $\frac{2\times5}{3\times5}$, $\frac{3\times4}{3\times5}$; $\frac{10}{15}$, $\frac{12}{15}$.

For 1, The fractions must become homony-

mous, since the denominator of each of the so changed fraction is the product of the denominators of the original fractions. The denominator of the first changed fraction is the product of the first denominator into the second, the denominator of the second changed fraction is the product of the second original denominator into the first; therefore each denominator of the changed fractions is the product of the same factors, and they must therefore be the same, that is, the fractions, are homonymous.

2. The fractions are not altered in their value, because we have multiplied the numerator and denominator of each by the denominator of the other, that is, by the same number (§ 8).

§ 12. Some abbreviations.

(*a*) If one denominator can be divided by the other without any remainder, it is only required to multiply the numerator and denominator of the fraction, which has the smaller denominator, by the quotient of the two denominators; viz., $\frac{5}{16}$ and $\frac{3}{8}$, $16 \div 8 = 2$, therefore we must multiply the numerator and denominator of $\frac{3}{8}$ into 2; $\frac{3 \times 2}{8 \times 2} = \frac{6}{16}$, and now $\frac{5}{16}$ and $\frac{6}{16}$ are homonymous.

(*b*) If the denominators of both can be divided by one and the same number, without a re-

mainder; viz., $\frac{5}{20}$ and $\frac{4}{12}$ are to be made homonymous; the denominators of both can be divided by 4, without any remainder; then the rule is, that we divide the denominator of each by this number, and the numerator and denominator of each fraction must be multiplied by the quotient which the division of that number in the other denominator gives; viz., $\frac{6}{20}$ and $\frac{3}{12}$, the numerators and denominators of $\frac{6}{20}$ must be multiplied by $12 \div 4 = 3$, that is $\frac{18}{60}$, and the numerator and denominator of $\frac{3}{12}$ by $20 \div 4 = 5$, that is, $\frac{3 \times 5}{12 \times 5} = \frac{15}{60}$; now are $\frac{6}{20}$ and $\frac{3}{12}$ made homonymous; viz., $\frac{18}{60}$ and $\frac{15}{60}$.

§ 13. More than two fractions are to be made homonymous, by multiplying the greatest denominator into all others, except those which can be divided into this greatest denominator, or into any other without a remainder, (this product we call the general denominator) by dividing each denominator into this general one, and multiplying each numerator into this quotient. For instance, if the fractions $\frac{3}{5}$, $\frac{4}{9}$, $\frac{2}{3}$, $\frac{5}{7}$ are to be made homonymous, we multiply 9, the greatest denominator, into 7 (63), and into 5 (315), but not into 3, because 3 can be divided into 9 without any remainder, then 315 is the general denominator;

5 into 315=63 ; this quotient multiplied into the numerator (3) of the denominator 5, gives 189, therefore, $\frac{3}{5}=\frac{189}{315}$ is the first fraction ; 9 into 315=35, $35\times4=140$, therefore $\frac{4}{9}=\frac{140}{315}$; 3 into 315=105, $105\times2=210$, therefore $\frac{210}{315}=\frac{2}{3}$; 7 into 315=45, $45\times5=225$, therefore $\frac{225}{315}=\frac{5}{7}$; thus we have $\frac{3}{5}$, $\frac{4}{9}$, $\frac{2}{3}$, $\frac{5}{7}$ changed into four homonymous fractions; viz., $\frac{189}{315}$, $\frac{140}{315}$, $\frac{210}{315}$, $\frac{225}{315}$.

The value of each is not altered, as we have the numerator and denominator of each multiplied by one and the same number.

§ 14. We can well compare fractions with named numbers. Since the numerator expresses a number of parts, this numerator we compare to the number of the named number ; since the denominator increased or decreased changes the value of each part, it can easily be compared to the name of the named number; for instance, 4*d*., 4*s*., £4, are three named numbers, each of which has another value after the name is changed; $\frac{4}{7}$, $\frac{4}{3}$, $\frac{4}{5}$ are three fractions, each of which has another value after its denominator is changed. Therefore the numerator can be compared with the number of the named numbers, the denominator with the name.

§ 15. Two fractions are to be added:

(*a*) If they are homonymous, by adding the

numerators, and writing under this sum the common denominators; viz., $\frac{3}{11}+\frac{4}{11}=\frac{7}{11}$; $\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$; for we can compare homonymous fractions to equal numbers, and as 2lb. + 5lb. = 7lb., in the same manner $\frac{2}{9}+\frac{5}{9}$ must be $\frac{7}{9}$.

(*b.*) If they are not homonymous, we make them homonymous, and add them in the same manner; viz., $\frac{5}{7}+\frac{2}{3}=\frac{15}{21}+\frac{14}{21}=\frac{29}{21}$; $\frac{3}{8}+\frac{4}{5}=\frac{15}{40}+\frac{32}{40}=\frac{47}{40}$, or we can say that we multiply each numerator by the denominator of the other fraction, and add these products, then this sum is the numerator of the sum, and we multiply their denominators, and this product is the denominator of the sum; viz., $\frac{3}{8}+\frac{4}{5}$; $3\times5+4\times8=15+32=47$, is the numerator of the sum, $8\times5=40$ is the denominator of the sum; therefore $\frac{3}{8}+\frac{4}{5}=\frac{47}{40}$.

Remarks.—(*a*) If we have to add two fractions, the numerators of which are 1, the sum of the denominators is the numerator, the product of the denominators is the denominator of the sum, viz. $\frac{1}{2}+\frac{1}{5}=\frac{7}{10}$, $2+5$ is the numerator, 2×5 is the denominator of the sum of both, $\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$.

The proof is evident, if we consider, that we ought to multiply each numerator into the denominator of the other; but, as here the numerators are equal, the denominators remain unchanged.

(*b*) Of course we can here use all those abbreviations before mentioned in § 12.

(*c*) If we have more than two fractions to add, we either add first two, and to this sum the third, and so on; or we make them homonymous by the general denominator, viz. after § 13, and add them as homonymous fractions must be added.

§ 16. Two fractions are to be subtracted:

(*a*) If they are homonymous, by subtracting the numerator, and writing under this remainder, which is the numerator, the common denominator, viz. $\frac{9}{11}-\frac{5}{11}=\frac{4}{11}$; for fractions can be compared to named numbers, and as 9 lbs. — 5lbs. = 4 lbs. so $\frac{9}{11}-\frac{5}{11}=\frac{4}{11}$.

(*b*) If they are not homonymous, by making them homonymous, and by subtracting them in the same manner, as in (*a*), viz. $\frac{5}{4}-\frac{3}{7}=\frac{35}{28}-\frac{12}{28}=\frac{23}{28}$; or we say that we multiply each numerator into the denominator of the other fraction, and subtract these products the one from the other, giving the common denominator as denominator to this difference, $\frac{5}{4}-\frac{3}{7}$, $5\cdot7-4\cdot3=21$, therefore $\frac{21}{28}$.

§ 17. Fractions are multiplied by multiplying the numerator into the numerator, and the denominator into the denominator, viz. $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$, $\frac{5}{6}\times\frac{7}{9}=\frac{35}{54}$.

We will here, instead of numbers, take letters, in order to prove it generally, for letters are of general signification for any number. $\frac{N}{D} \times \frac{n}{d}$ is after our rule $\frac{N \times n}{D \times d}$; for $\frac{n}{d}$ is after § 10 $n \div d$, therefore $\frac{N}{D} \times \frac{n}{d} = \frac{N}{D} \times n \div d$; but $\frac{N}{D} \times n = \frac{N \times n}{D}$ and this divided by $d = \frac{N \times n}{D \times d}$, which has to be proved, viz. that $\frac{N}{D} \times \frac{n}{d} = \frac{N \times n}{D \times d}$.

Now we will prove it in numbers. $\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5}$; for $\frac{4}{5} = 4 \div 5 . \therefore$* $\frac{2}{3} \times \frac{4}{5} = \frac{2}{3} \times 4 \div 5 = \frac{2 \times 4}{3} \div 5 = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$. Or in words :—If we have to multiply one fraction into another, and write instead of the other fraction the same value, viz., the numerator divided by the denominator, it is the same, as if we had first to multiply the fraction into the numerator, and then to divide it by the denominator of the other. That means, we have to multiply the numerator of the first fraction by the numerator of the other, and the denominator of the first by the denominator of the other; and that was the rule.

§ 18. Fractions are divided:

* $\therefore$ = therefore

(a) If they are homonymous, by dividing the numerators and dropping the equal denominator, viz. $\frac{12}{11} \div \frac{3}{11} = 4$, $\frac{20}{21} \div \frac{4}{21} = 5$.

For fractions can be compared to named numbers; and as, for instance, 5 lb. divided into 20 lb. is 4 times contained, not 4 lb. times, so must $\frac{5}{21}$, divided into $\frac{20}{21}$ be 4 times contained, not $\frac{4}{21}$ times.

(b) If the fractions are not homonymous, by inverting the divisor and multiplying one fraction into the other, viz. $\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \times \frac{6}{5} = \frac{12}{15}$, $\frac{3}{7} \div \frac{5}{9} = \frac{3}{7} \times \frac{9}{5} = \frac{27}{35}$. We ought properly to make the fractions homonymous, because fractions must, like named numbers, be homonymous, if we divide them; but we can prove by letters that our rule is only an abbreviation. For if $\frac{N}{D}$ were to be divided by $\frac{n}{d}$, and we would make them homonymous $\frac{N}{D} = \frac{N \times d}{D \times d}$, and $\frac{n}{d} = \frac{n \times D}{d \times D}$, therefore $\frac{N}{D} \div \frac{n}{d} = \frac{N \times d}{D \times d} \div \frac{n \times D}{d \times D} = \frac{N \times d}{n \times D}$. If we divide $\frac{N}{D}$ by $\frac{n}{d}$ after our rule, $\frac{N}{D} \times \frac{d}{n} = \frac{N \times d}{D \times n}$, (§ 17), we receive in both cases the same result; therefore our rule is only an abbreviation of the proper rule. In numbers we shall find the same, $\frac{2}{3} \div \frac{5}{7}$, if we make them homonymous, $\frac{16}{21} \div \frac{15}{21} = \frac{14}{15}$, and after

our rule $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$, in both cases there is the same result. We can also show whence this abbreviation. If we make the two fractions homonymous, we multiply the numerator and denominator of each fraction into the denominator of the other. If we only multiplied the numerator of each into the denominator of the other, we should only obtain the numerator of each; and we want here only the numerator, because the denominator is dropped if homonymous fractions are to be divided; therefore we have the product of the numerator of the first fraction into the denominator of the second, to be divided by the product of the numerator of the second into the denominator of the first; that is to invert the divisor and to multiply both.

§ 19. A mixed number we call a number, to which a fraction is to be added, only that we generally leave out the sign of addition, viz. $4\frac{1}{2} = 4 + \frac{1}{2}$. Performing this addition is called reducing the mixed number. A mixed number is to be reduced by multiplying the number into the denominator of the fraction, and adding to this product the numerator of the fraction; then this result is the numerator of the denominator of the fraction, viz. $4\frac{1}{2} = \frac{9}{2}$, $5\frac{3}{4} = \frac{23}{4}$. For if we

write 1 under the number, we have to add two fractions which are not homonymous; therefore we multiply the numerator of the first, which is that number, into the denominator of the second fraction, and add to it the product of the second numerator into the first denominator, which is 1; therefore we have only to add the second numerator, which is the same as our rule aforesaid.

§ 20. To subtract a fraction from a number is to be done in a very similar way; viz. we multiply the number by the denominator of the fraction, and subtract from it the numerator of the fraction, viz. $5 - \frac{4}{5} = \frac{25-4}{5} - \frac{21}{5}$.

The proof is just the same as in the preceding section, if we imagine, that 1 is written under the number. To multiply a fraction by a number is already shown in § 6.

§ 21. A number is to be divided by a fraction, by inverting the fraction, and multiplying both; $4 \div \frac{5}{7} = 4 \times \frac{7}{5} = \frac{28}{5}$, $4 \div \frac{7}{9} = \frac{36}{7}$. For if we think that 1 is written as denominator under the number, we have to divide two fractions after § 18, *b*; we must therefore invert the divisor and multiply; and since 1 does not multiply, we leave it out.

§ 22. We have already seen, in § 10, that a fraction is nothing but a division. We will here only remark, that in future we will always express a division by a fraction; and, on the contrary, use the division instead of a fraction, without any difference. For instance, instead of $36 \div 5$ we will say $\frac{36}{5}$, without any distinction, as there really is none. At the same time we see that the common expression, "we cannot divide a divisor into a smaller dividend," cannot in the least be applied as right; on the contrary, we see that the result of such a division is a proper fraction.

The Rule of Three.

§ 23. We call relation the comparison of two numbers with regard to their units. A geometrical relation (and this is the only one we have to consider here,) is that relation, in which we compare two numbers, so that we see, how many times the one is contained in the other; viz. 12 : 4 is a geometrical relation; therefore a geometrical relation is nothing but a division or a fraction.

That which stands as dividend we call here the first member, the divisor the second mem-

ber of the relation. The quotient is here also called exponent. We see that the exponent must be found by dividing the second member into the first.

§ 24. If the second member of a geometrical relation and the exponent are given, we find the first member by multiplying both; viz. if 3 be the second member, and 5 the exponent, the first is 15. For the second member of the relation is the divisor, the exponent the quotient, and we find the dividend from the divisor (3) and quotient (5) by multiplying both (15). If the first member and the exponent are given, we find the second member by dividing the exponent into the first member; viz. let 18 be the first member and 6 the exponent, then 3 is the second member; because we find, from the dividend (the first member) and the quotient, the divisor (second member), by dividing the quotient (6) into the dividend (18), that means here, the exponent into the first member of the relation (3).

§ 25. Relations are equal when their exponents are equal; viz. $6 : 2 = 9 : 3$, because the exponent of $6 : 2$ is 3, and the exponent of $9 : 3$ is $\frac{9}{3}$ or 3. Two equal relations connected are called a proportion; for instance, $6 : 2 :: 9 : 3$

is a proposition, and we read it: as 6 is to 2 so is 9 to 3.

The first and second member of the first relation are also called first and second member of the proportion. The first member of the second relation is called the third, and the second member of the second relation the fourth member of the proportion. The first and fourth members are also called the exterior, and the second and third members the interior members.

§ 26. If any three members of a proportion are given, we can find the fourth. For our purpose it will be sufficient to show, how we find from the three first members the fourth; viz. the fourth member of every real proportion is equal to the two interior members divided by the first. For instance, $8:4=6:3$, $3=\frac{4\times6}{8}$, $5:6=10:x$. If we call the fourth member x, x would be equal to $\frac{6\times10}{5}=12$. For the fourth member of a proportion is the second member of the second relation. The second member of a relation we find by dividing the exponent into the first. The first member of the second relation is the third of the proportion. This we have to divide by the exponent

of the second relation; but as we do not know the second member of it, we do not know its exponent. Now the exponent of the second relation is equal to that of the first, the exponent of the first is equal to a fraction, the numerator of which is the first member, the denominator the second member of the proportion. Therefore we have to divide the third member by this fraction; that means to invert it and multiply both. (§ 18, *b.*) Therefore we have to multiply the third member into a fraction, the numerator of which is the second member, and the denominator the first member; that means, we have to multiply the third member by the second member, and to give to this product the first member as denominator; that is, to divide this product by the first member.

§ 27. Most of our named numbers stand in geometrical proportion. (We will afterwards see which do, and which do not.) For instance, if 3 lb. cost 4*s.*, 6 lb. cost, of course, 8*s.* Now, as the 3 lb. are to 6 lbs. so are 4*s.* to 8*s.*, when the one ware is twice as much as the other, the price of the one is double that of the other. 3 lb. : 6 lb. : : 4*s.* : 8*s.* When the one ware would be three times greater or less than the other, the price of the one would be three times

greater or less than the price of the other. When 5 men dig 7 feet, 10 men would dig 14 feet; 5m. : 10 m. : : 7 ft. : 14 ft. When 3 horses consume 2 barrels of oats in a certain time, of course 12 horses, in the same time, and under the same conditions, will consume 8 barrels. If 1 acre of land costs £200, 5 acres will cost £1000, and so on. We can give this rule that all such named numbers stand in geome trical proportion, which are of this kind, that we receive the answer the more or the less, if the question is the more or the less. For instance, if 3 lb. cost 4*s*., 6 lb. cost 8*s*.; the more pounds, the more shillings they cost; the less pounds, the less shillings they cost. Five men dig 7 feet, 10 men dig 14 feet. These four named numbers stand in geometrical proportion; for the more men are digging, the more feet they will dig; the less are digging, the less feet they will dig. And so we see in the next example; the more barrels of oats, the more horses; the less horses, the less barrels of oats. We will afterwards see, what kind of proportion the other named numbers form.

§ 28. Since from three members of a proportion we can find the fourth, we can also from three named numbers of this sort (those which

stand in geometrical proportion) find the fourth, when we only write down the three given numbers so that they with the fourth would form a proportion. For this end let us first consider a little more such a proportion of named numbers. When 3 lbs. cost *5s.*, 6 lbs. cost *10s.* 3 lbs. : 6 lbs. : : *5s.* : *10s.* It is obvious,

1. That every relation can only contain homonymous members, since a relation is composed of two members, compared in regard to their units; and only homonymous things can be compared.

2. Four such named numbers are contained in two sentences; viz. when 3 lbs. cost *5s.*, one sentence, 6 lbs. cost *10s.*, the second sentence. Such four numbers stand in this position, that two are the conclusion or consequence of two others, which are the condition; that means, just as one number stands to another, so a third to a fourth;—as the 3 lbs. stand to *5s.* viz. 3 is the $\frac{3}{5}$ part of 5, just so is 6 the $\frac{3}{5}$ part of 10. In one word, the first and third member of such a proportion form the condition, the second and fourth the conclusion: 3 lbs. : 6 lbs. : : *5s.* to *10s.* When 3 lbs. cost *5s.*, first and third members are the condition, 6 lbs. cost *10s.* are the conclusion.

§ 29. The rule, how to find from three given numbers the fourth, is called the "Rule of Three," and is this :—The given number of the question is the second member, the homonymous one of the condition the first, the one left member of the condition the third, and x the fourth member of the proportion ; viz. when 3 lbs. cost $5s.$ how much will 6 lbs. cost? 6 is the number of the question, therefore the second member of the proportion ; 3 is the homonymous number of the condition, therefore the first member ; the other member of the condition is 5, the third ; and x the fourth member ; viz. $3 : 6 = 5 : x$, x therefore (§ 25) $= \frac{5 \times 6}{3} = 10$. Another example :—When 5 men dig 7 feet, how deep will 15 men dig? $5 : 15 = 7 : x$, $x = \frac{15 \times 7}{5} = 21$. When a locomotive goes 10 miles in 3 hours, how many miles will it go in 6 hours? $3 : 6 = 10 : x$, $x = \frac{6 \times 10}{3} = 20$ miles.

The evidence of this rule appears from § 28, viz.: suppose the fourth member of the proportion be known, and we call it x, then the homonymous number of the condition must be the third number, and since the second and fourth numbers, the first and third belong so

together, that the two former are the condition, the two latter the conclusion; if x be the fourth number, the other member of the question must be the second, as was stated in our rule.

§ 30. Examples:

1. When 5 lb. cost 3*s*., how much do 7 lb. cost?

5 lb. : 7 lb. : : 3*s*. : x *s*. Suppose 7 lb. cost x *s*., and let x be the fourth member of the proportion, then the homonymous member of the condition must be the third, viz. 3*s*.; and since the second and fourth member belong together, 7 lb., the number of the question, must be the second member, and 5 lb of course the first. Now, $x=\frac{3\times 7}{5}=\frac{21}{5}=4\frac{1}{5}$, and, $5 : 7 :: 3 : \frac{21}{5}$ is a right proportion; for the exponent of the first relation is $\frac{5}{7}$, that of the second $3 : \frac{21}{5}=3\times\frac{5}{21}=\frac{15}{21}=\frac{5}{7}$.

2. When 10 horses consume 14 bushels of oats in a certain time, how many will 7 horses consume in the same time?

$10 : 14 :: 7 : x$, $x=\frac{98}{10}=9\frac{4}{5}$ bushels.

3. When 25 yards of cloth are sufficient for 9 coats, how much is required for 4 coats?

$9 : 4 :: 25 : x$, $x=\frac{4\times 25}{9}=\frac{100}{9}=11\frac{1}{9}$.

4. If a man in 3 hours walks 16 miles, how many does he walk in 4 hours?

$3 : 4 :: 16 : x, \quad x = \frac{4 \times 16}{3} = \frac{64}{3} = 21\frac{1}{3}.$

§ 31. Since the fourth member is equal to the product of the second into the third divided by the first, and numerator and denominator can be divided by the same number without changing the value of the fraction, the first member and the product of the second and third can be divided by one and the same number without changing the proportion, viz.:

$3 : 15 :: 4 : x, \quad x = \frac{4.\ 15}{3} = \frac{4.\ \cancel{15}\ 5}{\cancel{3}\ 1}$; therefore, instead of $3 : 15 :: 4 : x$ we can say $\cancel{3}1 : \cancel{15}\,5 :: 4 : x$; thus we can divide the first and third members of a proportion by one and the same number. In the same manner it can be proved that we can divide the second and fourth member of a proportion by one and the same number, viz.,

$$4 : 5 = 12 : x, \text{ or } 1 : 5 = 3 : x.$$

Simple Interest.

§ 32. Interest-reckoning is nothing but the simple Rule of Three; it only appears to be different, viz., it seems, as if there were only two

numbers given, and we had to find the third; but in fact there are three numbers given, and we have to find the fourth. All the difference is in the names used in these reckonings, viz., the sum lent is called the capital or principal; interest, the sum given as premium or compensation for lending the capital; and per centage the interest of a hundred. It is this last name which causes occasionally some difficulty in understanding these accounts; we have only to observe, that per centage is the interest or profit upon a 100; if the 100 is of pounds, the per centage is also of pounds; if the 100 is of shillings, the per centage is of the same. And this expression per centage is not only the interest of 100, but any other profit or relation reckoned by a hundred, either profit or loss. There can be three cases of interest-accounts.

§ 33. (*a.*) If the capital and the per centage are given to find the interest; we have this proportion, as 100 is to the given capital, so is the per centage to x; for this proportion fully expressed would be thus: If 100 as capital gains so much interest, as the per centage expresses, how great is the interest of the given capital? The number of the question is the capital,

therefore the second member; 100 the homonymous number therefore, the first; and the per centage the third member of the proportion, viz., What is the interest of £250 at 4 per cent? (C may stand for capital, p Ct. for per centage, and I. for interest.) This would be fully expressed thus: If £100 C. gives £4 I., how great is the I. of £250 C.? Therefore $100 : 250 = 4 : x$, $x = 10$.

If we have to find the interest for more than one year, we first find it for one year, and then multiply this result by the number of years, viz.: How great are the interests of £250 to 4 pct. in 6 years?

$100 : 250 = 4 : x$, $x = 10$; therefore in 6 years $6 \times 10 =$ £60.

§ 34. (*b.*) If the interest and per centage are given, to find the capital; the proportion will be this: as the per centage is to the interest, so are 100 to x. For the question fully stated is this: When to that interest, which the per centage signifies, 100 is the capital, what will be the capital of the given interest? Therefore, the interests, the number of the question, must become the second number of the proportion, the homonymous number of the condition is the interest of 100 or per centage, therefore the first member, and 100 the third member of the

proportion, viz., What is the capital, which gains £14 I. at 4 p Ct.? 4 : 14 : : 100 : x, x=£350.

If the interest of more years than one is given, we must first find it for one year, by dividing the number of years into the I, and then we reckon in the same way, viz.: What is that capital which gives £14 interest at 4 p Ct. in 2 years; the I. of one year would be

$$\tfrac{14}{2}=£7$$

$$4 : 7 :: 100 : x$$

$$x=£175.$$

§ 35. If the capital and interest are given, to find the per centage, we name this proportion: The capital is to 100 as the interest is to x.

For this question fully expressed is this: if the given capital produces the given interest, how great is the I. of £100? viz., at what per centage is £420 lent, to bring £21 I.; or more fully stated, when £420 gives £21 I., how much will £100 give? 420 : 100=21 : x, x =5 p Ct.

If the interest for many years is given, we have of course only to find the interest for one year, by dividing the number of years into the interest, and then to do it after the same proportion; viz., at what per centage is £720 lent, to bring £21 I. in 2 years?

$$21 \div 2 = \frac{21}{2}$$
$$420 : 100 = \frac{21}{2} : x$$
$$x = 2\frac{1}{2}$$

Inverse Rule oj Three.

§ 36. We have mentioned before (§ 17), that all named numbers do not stand in such proportion, that if one increases, the other also increases, that more of the one requires more of the other, and the contrary. There are other named numbers which do not stand in such proportion; viz., if 5 men finish a piece of work in 6 days, 10 men of course will finish it in 3 days. Now, 5 m. : 10 m. : : 6 d. : 3 d. is not a right proportion, for the exponent of the first relation $\frac{5}{10}=\frac{1}{2}$, that of the second, $\frac{6}{3}=2$; but if we invert the members of the first relation, viz., 10 : 5 : : 6 : 3, the exponents beeome equal. These numbers are of that kind that more requires less, and less more: the more men work, the less days are required, and the less men work, the more days are required. We see that the proportion of each number must be so ordered, that if the exterior members are the condition, the interior are the conclusion and the contrary. Therefore, if three of such numbers are given, and we have to

find the fourth; the rule of three will be so far changed, that the number of the question must become the first number of the proportion; for if the fourth number be x, the homonymous number of the condition is the third, the other of the condition must be the second, and therefore the number of the question the first member of the proportion; viz., if 7 horses consume a certain quantity in 8 days, how long will the same food last for 2 horses? $2 : 7 = 8 : x$, $x = 28$. If £200 must be given upon interest, in order to bring at 4 p Ct. in a certain time a certain sum as interest, how great must be the principal, that it may in the same time bring the same interest, when it can be given only at 2½ p Ct.? The more p Ct. the less C. is required, therefore inverse rule of three.

$$\tfrac{5}{2} : 4 = 200 : x.$$

$$x = 800 \div \tfrac{5}{2} = \frac{800 \times 2}{5} = \frac{1600}{5} = £320.$$

The Rule on the Line.

§ 37. If the three given numbers stand in direct proportion, the fourth number is equal to the product of the number of the question multiplied by the inhomonymous number of the condition, and divided by the other member of the

condition; (after § 29) or rather the fourth number is equal to a fraction, of which the product of the two former numbers is the numerator, and the last number the denominator; or if instead of the horizontal line — we write a vertical one |, and consider the right side as numerator, and the left as denominator, the rule will be this: We write on the right side the number of the question, and the inhomonymous number of the condition, (or this number rather which is homonymous with the x), and on the left this number of the condition, which is homonymous with the number of the question. Then x is equal to the product of the right side divided by the left; for instance, if 3 lb. cost 5*s.*, how much does 7 lb. cost?

3 lb.—5*s.*	x *s.*	7 lbs.
7 lb.—?	3 lb.	5*s.*

$$x = \frac{35}{3} = 11\tfrac{2}{3}s.$$

If the numbers stand in inverse proportion, the number of the question is the first member of the proportion; therefore the rule on the line for such numbers is this—the two numbers of the condition must be written on the right side, and the given number of the question on the left; viz., when 5 men finish a piece of work in 7 days, how long will it require 10 men to do it?

$$\begin{array}{r|l} x \text{ d.} & 3 \text{ m} \\ 10 \text{ m.} & 7 \text{ d.} \end{array} \qquad x=\tfrac{35}{10}=3\tfrac{1}{2}.$$

The Double Rule of Three, or "*the Rule of Five.*"

§ 38. If five numbers are given, we can easily find the sixth by two proportions. For instance, when 7 men dig 200 feet in 9 days, how many feet do 5 men dig in 10 days?

7 m.— 9 d.—200 ft.
5 m.—10 d.—?

We first find the number of feet, supposing the number of days to be equal; viz. 7 men dig 200 feet in 9 days, how many feet will five men dig in the same time? $7 : 5 :: 200 : x$, $x=\frac{4\times200}{7}$ feet. And now we say, If 5 men dig in 9 days $\frac{5\times200}{7}$feet, how much will the same men dig in 10 days?

$$9 : 10 :: \frac{5\times200}{7} : x$$

$$x=\frac{10\times5\times200}{9\times7}=158\tfrac{40}{63} \text{ feet.}$$

If we write the last fraction for x; viz. $x=\frac{10\times5\times200}{9\times7}$, so that the numerator stands on

the right side of a vertical line, the denominator on the left, viz.

$$\begin{array}{c|c} & 10 \\ 9 & 4 \\ 7 & 200 \end{array}$$

we have the same rule as in § 37; viz. we write the given numbers of the question, and that of the condition, which is homonymous with x on the right side; on the left side, the two other numbers of the question. If some of the numbers stand in inverse proportion, their places must be changed; for instance, when 7 men dig a certain number of feet in 9 days, if they work 8 hours daily, how many men are required to finish the same work in 21 days, if they work 12 hours daily?

7 m.— 9 d.— 8 h.
? —21 d.—12 h.

After our rule it would be

$$\begin{array}{l|l} x\text{ m.} & 7\text{ m.} \\ 9\text{ d.} & 21\text{ d.} \\ 8\text{ h.} & 12\text{ h.} \end{array}$$

but now the days and hours stand in inverse proportion with the men; viz., the more men work, the less days are required, also the more men the less hours; therefore we must invert the numbers of days and hours; viz.,

$$\begin{array}{r|l} x \text{ m.} & 7 \text{ m.} \\ 21 \text{ d.} & 9 \text{ d.} \\ 12 \text{ h.} & 8 \text{ h.} \end{array}$$

and x is equal to $\frac{7 \cdot 9 \cdot 8}{21 \cdot 12}$, $x=2$ men.

The Rule of Many, or the compound Rule of Three. Regula Multiplex.*

§ 39. If more than five numbers are given, for instance, seven, nine, &c.; we find x in the same way either by proportion, or by the same rule on the line, viz., when 6 men dig in 3 weeks a piece of land which is 200 feet long and 40 broad, in how many weeks will 5 men finish their work, if they have to dig a piece of ground 250 feet long, and 20 feet broad?

6 m.—3 w.—200 ft. long—40 ft. broad.
5 m.— ? 250 „ —20 „

1. If 6 men dig a certain piece of land in 3 weeks, how long will it take 5 men to dig the same? This is inverse proportion, for the more men the less time is required, therefore, $5 : 6 = 3 : x$; $x = \frac{3 \times 6}{5}$.

2. If 5 men dig in $\frac{3 \times 6}{5}$ weeks 200 feet in

length, in how long will the same men dig 250 feet in length? $200 : 250 = \frac{3 \times 6}{5} : x$; $x = \frac{250 \times 3 \times 6.}{200 \times 5}$

3. If 5 men dig in $\frac{250 \times 5 \times 6}{200 \times 5}$ weeks, a piece of land which is 40 feet broad, how many weeks must the same men dig, if it is 20 feet broad? $40 : 20 = \frac{25 \times 3 \times 6}{200 \times 5} : x$; $x = \frac{20 \times 250 \times 3 \times 6}{40 \times 200 \times 5} = 2\frac{1}{4}$; or, when we use the rule on the line according to § 38, namely, the given numbers of the question, viz., 5 m., 250 ft. long, 20 ft. broad, and the homonymous numbers of x, 3 w., on the right side; the other numbers of the condition on the left side.

x w.	3 w.
6 m.	5 m.
200l.	250 l.
40 b.	20 b.

But before we multiply, we must invert those numbers, which stand in inverse proportion; the more men the less weeks are required, therefore inverse proportion; the others are in direct proportion, therefore we have now

x w.	3 w.
5 m.	6 m.
200 l.	250 l.
40 b.	20 b.

$$x=\frac{3\times6\times250\times20}{5\times200\times40}=2\tfrac{1}{4}.$$

§ 40. Since we can shorten every fraction by dividing numerator and denominator by the same number, we can do the same here, viz. :—

x w.	3 w.
$\cancel{5}$ m,	$\cancel{6}$ m. 3
$\cancel{4}$ $\cancel{2}\cancel{0}\cancel{0}$ l.	$\cancel{2}\cancel{5}\cancel{0}$l. $\cancel{5}\cancel{0}$.
$\cancel{2}$ $4\cancel{0}$ b.	$\cancel{2}\cancel{0}$ b.

$$x=\frac{3\,.\,3}{4}=\tfrac{9}{2}=2\tfrac{1}{4}.$$

We can put a denominator of a fraction from the right side as a factor to the left side, since the divisor is on this side; and a denominator from the left side as a factor to the right, the left side being the divisor, for a fraction as divisor, changes its denominator into a numerator, that means its denominator must be written on the right side. Therefore this rule briefly expressed would be thus :—In the reckoning on the line we can put every denominator of one side as factor (and as a whole number,) upon the other

side. So then all fractions are cut off. For instance, when 6 men finish a wall of $200\frac{1}{2}$ feet long, $20\frac{4}{5}$ feet high and $2\frac{1}{2}$ feet thick in 3 weeks, working 6 days in each week, and $10\frac{1}{2}$ hours a day, how many weeks must 8 men work on a wall which shall be 300 feet long, $30\frac{1}{4}$ feet high, $3\frac{1}{2}$ feet thick, if they only labour 5 days in the week, and $12\frac{2}{3}$ hours a day?

$$\begin{array}{ccccccc} \text{m.} & \text{ft. l.} & \text{ft. b.} & \text{ft. th.} & \text{w.} & \text{d.} & \text{h.} \\ 6- & 200\frac{1}{2}- & 20\frac{4}{5}- & 2\frac{1}{3}- & 3- & 6- & 10\frac{1}{2}. \\ 8- & 300\ - & 30\frac{1}{4}- & 3\frac{1}{2}- & ?- & 5- & 12\frac{2}{3}. \end{array}$$

First, we reduce all mixed numbers:

$$\begin{array}{ccccccc} \text{m.} & \text{l.} & \text{b.} & \text{th.} & \text{w.} & \text{d.} & \text{h.} \\ 6- & \frac{401}{2}- & \frac{104}{5}- & \frac{7}{3}- & 3- & 6- & \frac{21}{2}. \\ 8- & 300- & \frac{121}{4}- & \frac{7}{2}- & ?- & 5- & \frac{38}{3}. \end{array}$$

$$\begin{array}{r|l} x \text{ w.} & 3 \text{ w.} \\ 8\ \cancel{6}\text{m.} & \cancel{8}\text{ m. } 6. \\ \frac{401}{2} & 300 \text{ l.} \\ \frac{104}{5} & \frac{121}{4} \text{ h.} \\ \frac{7}{3} & \frac{7}{2} \text{ th.} \\ 5\ \cancel{6} & \cancel{6}\text{ d. } 6. \\ \frac{38\ \cancel{21}}{3\ \cancel{2}} & \frac{\cancel{38}}{\cancel{3}} \text{ hours} \frac{21}{2} \end{array}$$

The denominators as factors on the other side:

x w.	3 w.
8 m.	6 m.
$\frac{401}{\cancel{2}}$	300 l. 2.
$4\cdot\frac{104}{\cancel{8}}$	$\frac{121}{\cancel{4}}$ th . 5
$2\cdot\frac{7}{\cancel{3}}$	$\frac{7}{\cancel{2}}$ th . 3
5	6 d.
$2\cdot\frac{38}{\cancel{3}}$	$\frac{21}{\cancel{2}}$ h . 3.

$$x=\frac{648336150\cancel{0}}{88746112\cancel{0}}=7 \text{ weeks and about } \tfrac{1}{3}.$$

The Partners' Reckoning.

§ 41. Thus are those reckonings called, in which we have to divide a given number into a quantity of unequal parts, which yet must stand in a certain relation to one another. For instance, it is required to divide the number 16 into two such parts, that the one is related to the other, as 1 to 3. The rule is this: the sum of the relative numbers is to the number, which is to be divided, as each relative number to x; viz.

$$4 : 16=1 : x, \; x= 4$$
$$4 : 16=3 : x^1, x^1=12$$

Or thus,

$$4 : 16= \begin{cases} 1 : x \\ 3 : x^1 \end{cases}$$

The evidence appears easily hence, that when we reduce this reckoning to the Compound Rule of Three, so that we add the relative numbers, $(1+3=4,)$ we have these two proportions. If the number 4 were given to be divided into two such parts, that one is proportioned to the other, as 1 to 3, the first part of course would be 1, what must be the first part of the number 16?

$$4 : 16 = 1 : x$$

And when the number 4 is to be divided into two such parts, that the second one is 3, what is the second part of 16, divided in the same way?

$$4 : 16 = 3 : x^1$$

Or instead both.

$$4 : 16 = \begin{cases} 1 : x \\ 3 : x^1 \end{cases}$$

We now see the rule proved, and that the sum of the relative numbers is to the given number as each relative number to x; viz.

(1.) Two merchants have to part a profit of £200 2*s*. 6*d*., according to the sum, that each of them has given to the business.

A.	has given	£900 4*s*.
B.	„	£2000 5*s*.

$$A.\quad 900\tfrac{1}{5} = \frac{4501}{5}$$

$$B.\quad 2000\tfrac{1}{4} = \frac{8001}{4}$$

$$2900\tfrac{9}{20} = \frac{58009}{20}$$

$$\frac{8009}{20} : \frac{1601}{8} = \begin{cases} \frac{4501}{5} : x \\ \frac{8001}{4} : x^1 \end{cases}$$

$$x = \frac{4501.1601.\ \cancel{20}.4}{\underset{7}{\cancel{28}}\ .\ \cancel{8}\ .58009}$$

$$£62\ \frac{12985}{116018} = £62,\ 2s.\ \frac{95950}{116018}d.$$

$$x^1 = \frac{8001.1601.\ \cancel{20}5}{8.\ \cancel{4}.58009}$$

$$x^1 = £138 .\because. 7\frac{41007}{63009}d.$$

(2) A fortune of £4110 is to be divided between three heirs, according to their respective ages.

A. is 25 years old.
B. 27 „
C. 20 „

$$82 : \frac{16441}{4} = \begin{cases} 25 : x \\ 27 : x^1 \\ 30 : x^2 \end{cases}$$

$x\ = £1253\ \ 2s.$
$x^1 = £1333\ \ 7s.$
$x^2 = £1503\ 15s.$

$£4110\ \ 5s.$

(3) A company has to divide a profit of £400 amongst their four stockholders after their shares.

The share of A. is £200
„ B. „ £225
„ C. „ £350
„ D. „ £425

£1200

$$1200 : 400 = \begin{cases} 200 : x \\ 225 : x^1 \\ 350 : x^2 \\ 425 : x \end{cases}$$

x = £ 66 13s. 4d.
x^1 = £ 75 0s. 0d.
x^2 = £116 13s. 4d.
x^3 = £141 13s. 4d.

£400 0s. 0d.

(4) The stock of a bankrupt is £4500 4s. 6d. The three creditors, A., B., and C., have to demand these sums.

A. £2000 5s. 0d.
B. £3000 2s. 0d.
C. £4000 1s. 8d.

$$\frac{270013}{30} : \frac{270013}{60} \begin{cases} \frac{8001}{4} : x \\ \frac{30001}{10} : x^1 \\ \frac{48001}{12} : x^2 \end{cases}$$

$x = £1000\ 2s.\ 6d.$
$x^1 = £1500\ 1s.\ 0d.$
$x^2 = £2000\ 0s.\ 10d.$

$£4500\ 4s.\ 4d.$

Compound Partners' Reckoning.

§ 43. A Compound Partner Reckoning is that, in which we have to part a profit of £675, according to the sums and times, for which they have contributed to the business.

A. has given £500 for 3 years.
B. „ £600 „ 2 „

The rule is this:—We multiply the relative numbers, and reckon then, as if these products were simple relative numbers, viz.

$£500 \times 3$ y. $= 1500$
$£600 \times 2$ y. $= 1200$

Therefore $2700 : 675 = \begin{cases} 1500 : x \\ 1200 : x^1 \end{cases}$

Then $x = £375$, $x^1 = £300$

The evidence of this rule appears from the following reasoning. A. gave £500 for 3 years. If he would have the same profit, but only leave his money for one year, he must of course have had

to give £1500. And if B. would only have left his one year, and received the same profit, he must have given £1200. As now we have to divide £670 after one relation; for the time is now the same, viz. one year, as these two relations are changed into one by multiplying.

Examples of Compound Partner Reckoning, with the Rule of Three, and Compound Interest.

§ 44. (1) A company of four members has to divide a sum of £4073 6*s*. 8*d*., after the number of years each has given his capital for; they have all contributed the same capital, viz. £7833 6*s*. 8*d*.

A.	gave his for	3	years	4	months
B.	„	2	„	5	„
C.	„	5	„	4	„
D.	„	6	„	3	„

How much did each receive? and at what percentage had they given their money?

3	years	4	months
2	„	5	„
5	„	4	„
6	„	3	„
17	„	4	$= 17\frac{1}{3}$ y. $= \frac{52}{3}$

$$\frac{52}{3}:\frac{12220}{3}=\begin{cases}\frac{10}{3}:x\\ \frac{29}{12}:x^1\\ \frac{16}{3}:x^2\\ \frac{25}{4}:x^3\end{cases}$$

$x\ =$£ 783 6*s*. 8*d*.
$x^1=$£ 567 18*s*. 4*d*.
$x^2=$£1253 6*s*. 8*d*.
$x^3=$£1468 15*s*. 0*d*.

£4073 6*s*. 8*d*.

A. therefore has given a capital of £$\frac{23500}{3}$ and has in $\frac{10}{3}$ years received £$\frac{2350}{3}$, in one year $\frac{235}{3}$, therefore $\frac{23500}{3}:100=\frac{235}{3}:x$; $x=1$ per cent.

DECIMAL FRACTIONS.

§ 44. A decimal fraction is a fraction, the denominator of which is 1, with one or more cyphers to the right. A decimal fraction is therefore nothing but a fraction with a certain denominator; and the whole difference is this, that we express a decimal fraction in a shorter way, namely, we merely write the numerator, the denominator being signified by a comma or dot, cutting off as many figures on the right as there are cyphers in the denominator; viz. $\frac{6714}{100}=67,14$; $\frac{6714}{1000}=6,714$; $\frac{6714}{10000}=0,6714$; $\frac{6714}{100000}=0,06714$.

We see that, when the numerator has just as many figures, as the denominator has cyphers, we have to put a cypher to the left of the comma; and when the numerator has less figures than the denominator cyphers, we must put as many cyphers to the right of the comma, as there, are figures less; viz. $\frac{67}{10000}=0,0067$; $\frac{67}{100000}=0,00067$; $\frac{67}{10000000}=0,0000067$; &c.

We read the fraction either as a common fraction; viz. $0,067=\frac{67}{1000}$, 67 thousandths, or as a mixed number, if there are figures to the left of the comma; viz. $6,714=6\frac{714}{1000}$, or every figure by itself; namely, 0,714=7 tenths, 1 hundredth, 4 thousandths.

§ 45. A decimal fraction is not altered in its value by adding cyphers to the right of its figures; viz. 6,714=6,7140=6,714000, &c. For if we read the decimal fraction, after having added the cyphers, without regarding the comma, that is the numerator; we find that the numerator has as many more cyphers on the right, as we have added cyphers to the decimal fraction. That means, the numerator is multiplied by 1 and as many cyphers (1000...00). At the same time, we see, that the decimals are increased, that is, the cyphers of the denominator; or, the denominator is multiplied by 1 and as many cy-

phers. Therefore the numerator and the denominator are multiplied by the same number, and consequently the decimal fraction not altered in its value. (§ 8.)

§ 46. A decimal fraction is multiplied by 10, 100, 1000, &c., by placing the comma as many figures further to the right, as there are cyphers in the multiplicand; viz. $6,714 \times 10 = 67,14$; $6,714 \times 1000 = 6714,$; and of course, if there are more cyphers than figures, we leave the comma out, and add as many cyphers, as are required, viz. $6,714 \times 100000 = 67400$. For a fraction is multiplied by a number by dividing the number into the denominator. (§ 6.) The number is here, 10, 100, 1000, &c., the denominator: 10, 100, 1000; we therefore divide the number into the denominator, by cutting off as many cyphers from the denominator, as there are cyphers in that number. In other words, we diminish the decimals of the decimal fraction, or put the comma further to the right.

§ 47. We divide a decimal fraction by 10, 100, 1000, &c., by putting the comma as many figures to the left, as that number contains cyphers; viz. $671,4 \div 100 = 6,714$; $6,714 \div 1000 = 0,6714$; $6,714 \times 100000 = 0,00006714$, &c. For a fraction is divided by a number by multiplying that number into the denominator. (§ 7.)

The number is here, 10, 100, 1000, &c. We multiply the denominator, therefore, by that number, by adding as many cyphers to the right, as that number contains. That means, we increase the decimal so many times, or we put the comma as many figures to the left.

§ 48. Fractions are homonymous, when they have an equal number of decimals. We make inhomonymous decimal fractions homonymous, by adding as many cyphers to the right of those, which have less decimals, as are required to make the decimals of all equal; viz. 6,714; 51,4; 40,67 are thus made homonymous: 6,714; 51,400; 40,670; for it is obvious, that their value is not changed according to § 45.

§ 49. Decimal fractions are to be added and subtracted so: that the comma of one always stands directly under the comma of the other, and that we put the comma in the same place in the sum or remainder; viz. 87,506 + 6,74 + 356,3 + 0,01204 are written in this manner—

$$
\begin{array}{r@{,}l}
87 & 506 \\
6 & 74 \\
356 & 3 \\
0 & 01204 \\
\hline
450 & 55804
\end{array}
$$

Now we add them like whole numbers. That

is, without regarding the comma, and then we write the comma in the same place. An example for subtraction : 67,4514—4,125 is thus written down—

$$\begin{array}{r} 67{,}4514 \\ 4{,}125 \\ \hline 63{,}3264 \end{array}$$

The evidence of this rule easily appears, if we remember common fractions. We know that these can only be added or subtracted, when they are homonymous, and then we add or subtract the numerators, and give to this sum or remainder the common denominator. Now, decimal fractions are made homonymous by adding cyphers; and then we must add or subtract the numerators or decimal fractions, without regarding the comma. And since the cyphers can be neglected in adding numbers, we obtain the same sum, if we have written the decimal fractions with their commas one under the other, and added no cyphers to the right. It is evident, that we give the equal denominator to the sum or remainder of the numerators, when we put the comma in the same place.

REMARK.—In case the subtrahend has more

decimals than the minuend, we must of course, add cyphers to the latter, or at least, we must reckon, as if there were cyphers; for instance, 675,31 − 35,02067.

$$\begin{array}{r} 675,31000 \\ 35,02067 \\ \hline 640,28933 \end{array}$$

§ 50. Decimal fractions are to be multiplied like whole numbers, that is, by leaving out the comma, and giving as many decimals to the product, as there are decimals in all the factors together; viz. $35,67 \times 4,0405 = 144,124635$. For decimal fractions must be multiplied like other fractions, (§ 17,) by multiplying the numerator into the numerator, and the denominator into the denominator. The numerators here are the decimal fractions without a comma, and these we have multiplied; the denominators are the numbers 10, 100, 1000; if we multiply these, we receive 1 and as many cyphers, as there are together in the factors, or as many decimals, as there are together in the factors.

§ 51. Decimal fractions are to be divided by dividing them like whole numbers, without regarding the comma. But there are three cases

to be distinguished, after having finished the division of the divisor into the dividend, considered as whole numbers.

(*a*) If the divisor and dividend have an equal number of decimals, the quotient is a whole number ; that is, we have not to cut off any decimals from the quotient.

(*b*) If the divisor has more decimals than the dividend, the quotient is not only a whole number, but we must also add as many cyphers to the right, as the divisor has more decimals.

(*c*) If the divisor has fewer decimals than the dividend, we cut off as many decimals on the right, as it has less. This shall be explained by examples.

(*d*) 179,4048÷0,3504. We divide these decimal fractions, as if there were no commas.

```
3504)1794048(512
     17520
     -----
       4204
       3504
       ----
        7008
        7008
        ----
```

Therefore 179,4048÷0,3504=512.

(*b*) 68,32÷0,000305.

```
305)6832 (224
    610
    ---
     732
     610
     ---
     1220
     1220
     ----
```

As we have here added one cypher, (and this can be done without altering the value, according to § 45,) the dividend has at the end three decimals, the divisor six; therefore the quotient is 224000.

(*c*) 0,0006832÷3,05.

```
305)6832 (224
    610
    ---
     732
     610
     ---
     1220
     1220
     ----
```

Now, the dividend has eight decimals and the

divisor two, at the end of the reckoning ; therefore we have to cut off six decimals from the quotient 224, making it 0,000224. The proof will easily be found, if we compare this method of reckoning given and proved in common fractions :

In the first case, where the decimals are equal, the decimal fractions are homonymous. Homonymous fractions are to be divided by dividing the numerator only, and dropping the denominator, &c. (§ 18, *a.*) The numerators are the decimal fractions without the comma. The denominators are dropped, if we do not cut off any decimals, that is, if we consider the quotient of the numerators as a whole number.

(*b*) If the divisor has more decimals than the dividend, the decimal fractions are inhomonymous ; and we must therefore (by § 18, *b*) invert the divisor, and multiply both. Let us imagine the decimal fractions written as common fractions, and we can express it in the following manner :—

$$\frac{\text{Figures}}{1\cancel{0}\cancel{0}\cancel{0}} \times \frac{1000\cancel{0}\cancel{0}\cancel{0}}{\text{Figures}}$$

We see the figures of the dividend are in the numerator, its cyphers in the denominator ; the

figures of the divisor are in the denominator, its cyphers in the numerator. Now the divisor has more decimals, therefore more cyphers; and if we divide the denominator of the dividend into the inverted numerator of the divisor, there remain 1 and as many cyphers in the numerator, as the divisor had more decimals. Therefore, if we divide the figures of the divisor into that of the denominator, we have to multiply this quotient by that remainder, that is, to add as many cyphers on the right.

(*c*) If the denominator has more decimals, and we invert the divisor, we obtain the remainder of the cyphers in the denominator. Thus—

$$\frac{\text{Figures}}{10000\not{0}\not{0}\not{0},\ \&c.} \times \frac{1\not{0}\not{0}\not{0},\ \&c.}{\text{Figures}}$$

And after having divided the figures of the divisor into those of the denominator, we have this quotient to divide by 1 and the remainder of the cyphers; in other words, to cut off as many decimals, as have remained, or as the dividend has more decimals.

§ 52. Since a decimal fraction is not altered in its value by adding cyphers to its right: we can add as many, as we like, when we divide a

decimal fraction by another fraction or number, and there is a remainder; viz., 64,05÷0,375 thus :

```
0,375)64,0500(170,8
       37,5
       ----
       2655
       2625
       ----
         3000
         3000
         ----
```

But we may sometimes divide as long as we like, yet never come to an end; and then this rule is to be observed: we are to stop, when it causes no fault in the reckoning. For instance, if we are reckoning with shillings, it causes no fault, (at least none of any consequence,) if we leave out all the decimals after the fourth; for $\frac{1}{100000}$ of a shilling is of a very trifling value. But when we stop in dividing, we have to find the next decimal, (which we intend to leave out,) and to add 1 to the last number, if this found decimal is greater than 5; to add nothing if it is less than 5. When it is 5, we can either add or not. It is better, however, to add 1, if figures, and not cyphers, follow the 5. This rule is, to

make the fault as small as possible. For example, if we intend to cease dividing, after having found the fourth decimal in the quotient, we must first find the fifth. Suppose the fifth decimal to be 8, we must add 1 to the last decimal, that is the fourth: this 1 is equal to $\frac{1}{10000}$; but $\frac{8}{100000}$ is less than $\frac{1}{10000}$, the difference is $\frac{2}{100000}$; but if we did not add 1, and left the $\frac{8}{100000}$, we should make a fault of $\frac{8}{100000}$. Thus we see that, if we follow this rule in other cases, we cause a smaller fault.

§ 53. A common fraction is to be changed into a decimal fraction, by writing a comma after its numerator, and after it as many cyphers, as we want, and then we divide the denominator into this numerator; viz. $\frac{3}{4}=\frac{3,00}{4}=0,75$; $\frac{5}{6}=\frac{5,000}{6}=0,833$,&c. Those fractions, which have no end, are called irradical decimals, and when the same decimals are repeated periodical decimal fractions; as 0,83333 or 0,8360630 &c.

I have added these few paragraphs of decimals, as even in common life their practice is most useful. For we know, that it is shorter to reckon sometimes with mixed numbers, and sometimes with fractions. In decimals we have both continually. Besides we can use many abbreviations in decimals, which we cannot do in common fractions.

This theory of common arithmetic can of course only prove its advantages when fully practised. For children who, in consequence of their youth, are unable to understand the demonstrations of the rules here given, the rules at least may be used, and will, it is hoped, be found more beneficial than most of those contained in other works of this description.

FINIS.

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